Gröbner bases of modules over $\sigma - PBW$ extensions

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Abstract

For $\sigma - PWB$ extensions, we extend to modules the theory of Gröbner bases of left ideals presented in [5]. As an application, if A is a bijective quasi-commutative $\sigma - PWB$ extension, we compute the module of syzygies of a submodule of the free module A^m.

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1 Introduction

In this paper we present the theory of Gröbner bases for submodules of A^m, $m \geq 1$, where $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a $\sigma - PBW$ extension of R, with R a LGS ring (see Definition 12) and $Mon(A)$ endowed with some monomial order (see Definition 9). A^m is the left free A-module of column vectors of length $m \geq 1$; if A is bijective, A is a left Noetherian ring (see [8]), then A is an IBN ring (Invariant Basis Number), and hence, all bases of the free module A^m have m elements. Note moreover that A^m is a left Noetherian, and hence, any submodule of A^m is finitely generated. The main purpose is to define and calculate Gröbner bases for submodules of A^m, thus, we will define the monomials in A^m, orders on the monomials, the concept of reduction, we will construct a Division Algorithm, we will give equivalent conditions in order to define Gröbner bases, and finally, we will compute Gröbner bases using a procedure similar to Buchberger's Algorithm in the particular case of quasi-commutative bijective $\sigma - PBW$ extensions. The results presented here generalize those of [5] where σ-PBW extensions were defined and the theory of Gröbner bases for the left ideals was constructed. Most of proofs are easily adapted from [5] and hence we will omit them. As an application, the final section of the paper concerns with the computation of the module of syzygies of a given submodule of A^m for the particular case when A is bijective quasi-commutative.

Definition 1. *Let R and A be rings, we say that A is a $\sigma - PBW$ extension of R (or skew PBW extension), if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finite elements $x_1, \ldots, x_n \in A - R$ such A is a left R-free module with basis*

$$Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case we say also that A is a left polynomial ring over R with respect to $\{x_1, \ldots, x_n\}$ and $Mon(A)$ is the set of standard monomials of A. Moreover, $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r - c_{i,r} x_i \in R. \tag{1.1}$$

(iv) *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.2}$$

Under these conditions we will write $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$.

The following proposition justifies the notation that we have introduced for the skew PBW extensions.

Proposition 2. *Let A be a $\sigma{-}PBW$ extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [5]. □

A particular case of $\sigma - PBW$ extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective. We have the following definition.

Definition 3. *Let A be a $\sigma - PBW$ extension.*

(a) *A is quasi-commutative if the conditions (iii) and (iv) in the Definition 1 are replaced by*

  (iii') *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r = c_{i,r} x_i. \tag{1.3}$$

  (iv') *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.4}$$

(b) *A is bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.*

Some interesting examples of $\sigma - PBW$ extensions were given in [5]. We repeat next some of them without details.

Example 4. (i) Any PBW extension (see [2]) is a bijective $\sigma{-}PBW$ extension.
(ii) Any skew polynomial ring $R[x;\sigma,\delta]$, with σ injective, is a $\sigma - PBW$ extension; in this case we have $R[x;\sigma,\delta] = \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x;\sigma]$ is quasi-commutative.
(iii) Any iterated skew polynomial ring $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a $\sigma - PBW$ extension if it satisfies the following conditions:

For $1 \leq i \leq n$, σ_i is injective.
For every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$.
For $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$, and c has a left inverse.
For $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_i$.

Under these conditions we have

$$R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] = \sigma(R)\langle x_1,\ldots,x_n\rangle.$$

In particular, any Ore algebra $K[t_1,\ldots,t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ (K a field) is a $\sigma - PBW$ extension if it satisfies the following condition:

For $1 \leq i \leq n$, σ_i is injective.

Some concrete examples of Ore algebras of injective type are the following.
The algebra of shift operators: let $h \in K$, then the algebra of shift operators is defined by $S_h := K[t][x_h; \sigma_h, \delta_h]$, where $\sigma_h(p(t)) := p(t-h)$, and $\delta_h := 0$ (observe that S_h can be considered also as a skew polynomial ring of injective type). Thus, S_h is a quasi-commutative bijective $\sigma-PBW$ extension.
The mixed algebra D_h: let again $h \in K$, then the mixed algebra D_h is defined by $D_h := K[t][x; i_{K[t]}, \frac{d}{dt}][x_h; \sigma_h, \delta_h]$, where $\sigma_h(x) := x$. Then, D_h is a quasi-commutative bijective $\sigma-PBW$ extension.
The algebra for multidimensional discrete linear systems is defined by $D := K[t_1, \dots, t_n][x_1, \sigma_1, 0] \cdots [x_n; \sigma_n, 0]$, where

$$\sigma_i(p(t_1, \dots, t_n)) := p(t_1, \dots, t_{i-1}, t_i + 1, t_{i+1}, \dots, t_n), \ \sigma_i(x_i) = x_i, \ 1 \leq i \leq n.$$

D is a quasi-commutative bijective $\sigma-PBW$ extension.
(iv) Additive analogue of the Weyl algebra: let K be a field, the K-algebra $A_n(q_1, \dots, q_n)$ is generated by $x_1, \dots, x_n, y_1, \dots, y_n$ and subject to the relations:

$$\begin{gathered} x_j x_i = x_i x_j, y_j y_i = y_i y_j, \ 1 \leq i, j \leq n, \\ y_i x_j = x_j y_i, \ i \neq j, \\ y_i x_i = q_i x_i y_i + 1, \ 1 \leq i \leq n, \end{gathered}$$

where $q_i \in K - \{0\}$. $A_n(q_1, \dots, q_n)$ satisfies the conditions of (iii) and is bijective; we have

$$A_n(q_1, \dots, q_n) = \sigma(K[x_1, \dots, x_n])\langle y_1, \dots, y_n \rangle.$$

(v) Multiplicative analogue of the Weyl algebra: let K be a field, the K-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1, \dots, x_n$ and subject to the relations:

$$x_j x_i = \lambda_{ji} x_i x_j, \ 1 \leq i < j \leq n,$$

where $\lambda_{ji} \in K - \{0\}$. $\mathcal{O}_n(\lambda_{ji})$ satisfies the conditions of (iii), and hence

$$\mathcal{O}_n(\lambda_{ji}) = \sigma(K[x_1])\langle x_2, \dots, x_n \rangle.$$

Note that $\mathcal{O}_n(\lambda_{ji})$ is quasi-commutative and bijective.
(vi) q-Heisenberg algebra: let K be a field , the K-algebra $h_n(q)$ is generated by $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n$ and subject to the relations:

$$\begin{gathered} x_j x_i = x_i x_j, z_j z_i = z_i z_j, y_j y_i = y_i y_j, \ 1 \leq i, j \leq n, \\ z_j y_i = y_i z_j, z_j x_i = x_i z_j, y_j x_i = x_i y_j, \ i \neq j, \\ z_i y_i = q y_i z_i, z_i x_i = q^{-1} x_i z_i + y_i, y_i x_i = q x_i y_i, \ 1 \leq i \leq n, \end{gathered}$$

with $q \in K - \{0\}$. $h_n(q)$ is a bijective $\sigma-PBW$ extension of K:

$$h_n(q) = \sigma(K)\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n \rangle.$$

(vi) Many other examples are presented in [8].

Definition 5. *Let A be a $\sigma - PBW$ extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.*

(i) *For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.*

(ii) *For $X = x^\alpha \in Mon(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.*

(iii) *Let $0 \neq f \in A$, $t(f)$ is the finite set of terms that conform f, i.e., if $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1X_1, \ldots, c_tX_t\}$.*

(iv) *Let f be as in* (iii), *then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.*

The $\sigma - PBW$ extensions can be characterized in a similar way as was done in [3] for PBW rings.

Theorem 6. *Let A be a left polynomial ring over R w.r.t $\{x_1, \ldots, x_n\}$. A is a $\sigma - PBW$ extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in Mon(A)$ and every $0 \neq r \in R$ there exists unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{1.5}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(b) *For every $x^\alpha, x^\beta \in Mon(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.6}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Proof. See [5]. □

Remark 7. (i) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.
(ii) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$, then we it is easy to check the following identities:

$$\begin{aligned} \sigma^\theta(c_{\gamma,\beta}) c_{\theta,\gamma+\beta} &= c_{\theta,\gamma} c_{\theta+\gamma,\beta}, \\ \sigma^\theta(\sigma^\gamma(c)) c_{\theta,\gamma} &= c_{\theta,\gamma} \sigma^{\theta+\gamma}(c). \end{aligned}$$

(iii) We observe if A is a $\sigma - PBW$ extension quasi-commutative, then from the proof of Theorem 6 (see [5]) we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$, for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$.
(iv) We have also that if A is a bijective $\sigma - PBW$ extension, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

A key property of σ-PBW extensions is the content of the following theorem.

Theorem 8. *Let A be a bijective skew PBW extension of R. If R is a left Noetherian ring then A is also a left Noetherian ring.*

Proof. See [8]. $\square$

Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a $\sigma - PBW$ extension of R and let $\succeq$ be a total order defined on $Mon(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$ we will write $x^\alpha \succ x^\beta$. Let $f \neq 0$ be a polynomial of A, if

$$f = c_1X_1 + \cdots + c_tX_t,$$

with $c_i \in R - \{0\}$ and $X_1 \succ \cdots \succ X_t$ are the monomials of f, then $lm(f) := X_1$ is the *leading monomial* of f, $lc(f) := c_1$ is the *leading coefficient* of f and $lt(f) := c_1X_1$ is the *leading term* of f. If $f = 0$, we define $lm(0) := 0, lc(0) := 0, lt(0) := 0$, and we set $X \succ 0$ for any $X \in Mon(A)$. Thus, we extend $\succeq$ to $Mon(A) \cup \{0\}$.

Definition 9. *Let $\succeq$ be a total order on $Mon(A)$, we say that $\succeq$ is a monomial order on $Mon(A)$ if the following conditions hold:*

(i) *For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in Mon(A)$*

$$x^\beta \succeq x^\alpha \Rightarrow lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda).$$

(ii) *$x^\alpha \succeq 1$, for every $x^\alpha \in Mon(A)$.*

(iii) *$\succeq$ is degree compatible, i.e., $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.*

Monomial orders are also called *admissible orders*. From now on we will assume that $Mon(A)$ is endowed with some monomial order.

Definition 10. *Let $x^\alpha, x^\beta \in Mon(A)$, we say that x^α divides x^β, denoted by $x^\alpha | x^\beta$, if there exists $x^\gamma, x^\lambda \in Mon(A)$ such that $x^\beta = lm(x^\gamma x^\alpha x^\lambda)$.*

Proposition 11. *Let $x^\alpha, x^\beta \in Mon(A)$ and $f, g \in A - \{0\}$. Then,*

(a) *$lm(x^\alpha g) = lm(x^\alpha lm(g)) = x^{\alpha + \exp(lm(g))}$. In particular,*

$$lm(lm(f)lm(g)) = x^{\exp(lm(f)) + \exp(lm(g))}$$

and

$$lm(x^\alpha x^\beta) = x^{\alpha+\beta}. \tag{1.7}$$

(b) *The following conditions are equivalent:*

(i) *$x^\alpha | x^\beta$.*

(ii) *There exists a unique $x^\theta \in Mon(A)$ such that $x^\beta = lm(x^\theta x^\alpha) = x^{\theta+\alpha}$ and hence $\beta = \theta + \alpha$.*

(iii) *There exists a unique $x^\theta \in Mon(A)$ such that $x^\beta = lm(x^\alpha x^\theta) = x^{\alpha+\theta}$ and hence $\beta = \alpha + \theta$.*

(iv) $\beta_i \geq \alpha_i$ *for* $1 \leq i \leq n$, *with* $\beta := (\beta_1, \dots, \beta_n)$ *and* $\alpha := (\alpha_1, \dots, \alpha_n)$.

Proof. See [5]. □

We note that a least common multiple of monomials of $Mon(A)$ there exists: in fact, let $x^\alpha, x^\beta \in Mon(A)$, then $lcm(x^\alpha, x^\beta) = x^\gamma \in Mon(A)$, where $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i := \max\{\alpha_i, \beta_i\}$ for each $1 \leq i \leq n$.

Some natural computational conditions on R will be assumed in the rest of this paper (compare with [7]).

Definition 12. *A ring R is left Gröbner soluble (LGS) if the following conditions hold:*

(i) *R is left Noetherian.*

(ii) *Given $a, r_1, \dots, r_m \in R$ there exists an algorithm which decides whether a is in the left ideal $Rr_1 + \cdots + Rr_m$, and if so, find $b_1, \dots, b_m \in R$ such that $a = b_1 r_1 + \cdots + b_m r_m$.*

(iii) *Given $r_1, \dots, r_m \in R$ there exists an algorithm which finds a finite set of generators of the left R-module*

$$Syz_R[r_1 \ \cdots \ r_m] := \{(b_1, \dots, b_m) \in R^m | b_1 r_1 + \cdots + b_m r_m = 0\}.$$

The three above conditions imposed to R are needed in order to guarantee a Gröbner theory in the rings of coefficients, in particular, to have an effective solution of the membership problem in R (see (ii) in Definition 20 below). From now on we will assume that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a $\sigma - PBW$ extension of R, where R is a LGS ring and $Mon(A)$ is endowed with some monomial order.

We conclude this chapter with a remark about some other classes of noncommutative rings of polynomial type close related with σ-PBW extensions.

Remark 13. (i) Viktor Levandovskyy has defined in [6] the G-algebras and has constructed the theory of Gröbner bases for them. Let K be a field, a K-algebra A is called a *G-algebra* if $K \subset Z(A)$ (center of A) and A is generated by a finite set $\{x_1, \dots, x_n\}$ of elements that satisfy the following conditions: (a) the collection of standard monomials of A, $Mon(A) = Mon(\{x_1, \dots, x_n\})$, is a K-basis of A. (b) $x_j x_i = c_{ij} x_i x_j + d_{ij}$, for $1 \leq i < j \leq n$, with $c_{ij} \in K^*$ and $d_{ij} \in A$. (c) There exists a total order $<_A$ on $Mon(A)$ such that for $i < j$, $lm(d_{ij}) <_A x_i x_j$. (d) For $1 \leq i < j < k \leq n$, $c_{ik} c_{jk} d_{ij} x_k - x_k d_{ij} + c_{jk} x_j d_{ik} - c_{ij} d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} x_i d_{jk} = 0$. According to this definition, the coefficients of a polynomial in a G-algebra are in a field and they commute with the variables $x_1, \dots, x_n$. From this, and also from (c) and (d), we conclude that the class of G-algebras does not coincide with the class of σ-PBW extensions. However, the intersection of these two classes of rings is not empty. In fact, the universal enveloping algebra of a finite dimensional Lie algebra, Weyl algebras and the

additive or multiplicative analogue of a Weyl algebra, are G-algebras and also σ-PBW extensions.
(ii) A similar remark can be done with respect to PBW rings and algebras defined by Bueso, Gómez-Torrecillas and Verschoren in [4].

2 Monomial orders on $Mon(A^m)$

We will often write the elements of A^m also as row vectors if this not represent confusion. We recall that the canonical basis of A^m is

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \boldsymbol{e}_m = (0, 0, \ldots, 1).$$

Definition 14. *A monomial in A^m is a vector $\boldsymbol{X} = X\boldsymbol{e}_i$, where $X = x^\alpha \in Mon(A)$ and $1 \leq i \leq m$, i.e.,*

$$\boldsymbol{X} = X\boldsymbol{e}_i = (0, \ldots, X, \ldots, 0),$$

where X is in the i-th position, named the index of $\boldsymbol{X}$, $ind(\boldsymbol{X}) := i$. A term is a vector $c\boldsymbol{X}$, where $c \in R$. The set of monomials of A^m will be denoted by $Mon(A^m)$. Let $\boldsymbol{Y} = Y\boldsymbol{e}_j \in Mon(A^m)$, we say that $\boldsymbol{X}$ divides $\boldsymbol{Y}$ if $i = j$ and X divides Y. We will say that any monomial $\boldsymbol{X} \in Mon(A^m)$ divides the null vector $\boldsymbol{0}$. The least common multiple of $\boldsymbol{X}$ and $\boldsymbol{Y}$, denoted by $lcm(\boldsymbol{X}, \boldsymbol{Y})$, is $\boldsymbol{0}$ if $i \neq j$, and $U\boldsymbol{e}_i$, where $U = lcm(X, Y)$, if $i = j$. Finally, we define $\exp(\boldsymbol{X}) := \exp(X) = \alpha$ and $\deg(\boldsymbol{X}) := \deg(X) = |\alpha|$.

We now define monomials orders on $Mon(A^m)$.

Definition 15. *A monomial order on $Mon(A^m)$ is a total order $\succeq$ satisfying the following three conditions:*

(i) *$lm(x^\beta x^\alpha)\boldsymbol{e}_i \succeq x^\alpha \boldsymbol{e}_i$, for every monomial $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i \in Mon(A^m)$ and any monomial x^β in $Mon(A)$.*

(ii) *If $\boldsymbol{Y} = x^\beta \boldsymbol{e}_j \succeq \boldsymbol{X} = x^\alpha \boldsymbol{e}_i$, then $lm(x^\gamma x^\beta)\boldsymbol{e}_j \succeq lm(x^\gamma x^\alpha)\boldsymbol{e}_i$ for all $\boldsymbol{X}, \boldsymbol{Y} \in Mon(A^m)$ and every $x^\gamma \in Mon(A)$.*

(iii) *$\succeq$ is degree compatible, i.e., $\deg(\boldsymbol{X}) \geq \deg(\boldsymbol{Y}) \Rightarrow \boldsymbol{X} \succeq \boldsymbol{Y}$.*

If $\boldsymbol{X} \succeq \boldsymbol{Y}$ but $\boldsymbol{X} \neq \boldsymbol{Y}$ we will write $\boldsymbol{X} \succ \boldsymbol{Y}$. $\boldsymbol{Y} \preceq \boldsymbol{X}$ means that $\boldsymbol{X} \succeq \boldsymbol{Y}$.

Proposition 16. *Every monomial order on $Mon(A^m)$ is a well order.*

Proof. We can easy adapt the proof for left ideals presented in [5]. □

Given a monomial order $\succeq$ on $Mon(A)$, we can define two natural orders on $Mon(A^m)$.

Definition 17. *Let $\boldsymbol{X} = X\boldsymbol{e}_i$ and $\boldsymbol{Y} = Y\boldsymbol{e}_j \in Mon(A^m)$.*

(i) *The TOP (term over position) order is defined by*

$$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ or \\ X = Y \, and \quad i > j. \end{cases}$$

(ii) *The TOPREV order is defined by*

$$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ or \\ X = Y \, and \quad i < j. \end{cases}$$

Remark 18. (i) Note that with TOP we have

$$\boldsymbol{e}_m \succ \boldsymbol{e}_{m-1} \succ \cdots \succ \boldsymbol{e}_1$$

and

$$\boldsymbol{e}_1 \succ \boldsymbol{e}_2 \succ \cdots \succ \boldsymbol{e}_m$$

for TOPREV.
(ii) The POT (position over term) and POTREV orders defined in [1] and [7] for modules over classical polynomial commutative rings are not degree compatible.
(iii) Other examples of monomial orders in $Mon(A^m)$ are considered in [4].

We fix monomial orders on $Mon(A)$ and $Mon(A^m)$; let $\boldsymbol{f} \neq \mathbf{0}$ be a vector of A^m, then we may write $\boldsymbol{f}$ as a sum of terms in the following way

$$\boldsymbol{f} = c_1 \boldsymbol{X}_1 + \cdots + c_t \boldsymbol{X}_t,$$

where $c_1, \dots, c_t \in R - \{0\}$ and $\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \cdots \succ \boldsymbol{X}_t$ are monomials of $Mon(A^m)$.

Definition 19. *With the above notation, we say that*

(i) $lt(\boldsymbol{f}) := c_1 \boldsymbol{X}_1$ *is the leading term of* $\boldsymbol{f}$.

(ii) $lc(\boldsymbol{f}) := c_1$ *is the leading coefficient of* $\boldsymbol{f}$.

(iii) $lm(\boldsymbol{f}) := \boldsymbol{X}_1$ *is the leading monomial of* $\boldsymbol{f}$.

For $\boldsymbol{f} = \mathbf{0}$ we define $lm(\mathbf{0}) = \mathbf{0}, lc(\mathbf{0}) = 0, lt(\mathbf{0}) = \mathbf{0}$, and if $\succeq$ is a monomial order on $Mon(A^m)$, then we define $\mathbf{X} \succ \mathbf{0}$ for any $\mathbf{X} \in Mon(A^m)$. So, we extend $\succeq$ to $Mon(A^m) \cup \{\mathbf{0}\}$.

3 Reduction in A^m

The reduction process in A^m is defined as follows.

Definition 20. *Let F be a finite set of non-zero vectors of A^m, and let $\boldsymbol{f}, \boldsymbol{h} \in A^m$, we say that $\boldsymbol{f}$ reduces to $\boldsymbol{h}$ by F in one step, denoted $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$, if there exist elements $\boldsymbol{f}_1, \dots, \boldsymbol{f}_t \in F$ and $r_1, \dots, r_t \in R$ such that*

(i) $lm(\boldsymbol{f}_i)|lm(\boldsymbol{f})$, $1 \leq i \leq t$, *i.e.,* $ind(lm(\boldsymbol{f}_i)) = ind(lm(\boldsymbol{f}))$ *and there exists* $x^{\alpha_i} \in Mon(A)$ *such that* $\alpha_i + \exp(lm(\boldsymbol{f}_i)) = \exp(lm(\boldsymbol{f}))$.

(ii) $lc(\boldsymbol{f}) = r_1\sigma^{\alpha_1}(lc(\boldsymbol{f}_1))c_{\alpha_1,\boldsymbol{f}_1} + \cdots + r_t\sigma^{\alpha_t}(lc(\boldsymbol{f}_t))c_{\alpha_t,\boldsymbol{f}_t}$, *with* $c_{\alpha_i,\boldsymbol{f}_i} := c_{\alpha_i,\exp(lm(\boldsymbol{f}_i))}$.

(iii) $\boldsymbol{h} = \boldsymbol{f} - \sum_{i=1}^{t} r_i x^{\alpha_i}\boldsymbol{f}_i$.

We say that $\boldsymbol{f}$ *reduces to* $\boldsymbol{h}$ *by* F, *denoted* $\boldsymbol{f} \xrightarrow{F}_+ \boldsymbol{h}$, *if and only if there exist vectors* $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{t-1} \in A^m$ *such that*

$$\boldsymbol{f} \xrightarrow{\quad F \quad} \boldsymbol{h}_1 \xrightarrow{\quad F \quad} \boldsymbol{h}_2 \xrightarrow{\quad F \quad} \cdots \xrightarrow{\quad F \quad} \boldsymbol{h}_{t-1} \xrightarrow{\quad F \quad} \boldsymbol{h}.$$

$\boldsymbol{f}$ *is reduced (also called minimal) w.r.t.* F *if* $\boldsymbol{f} = \boldsymbol{0}$ *or there is no one step reduction of* $\boldsymbol{f}$ *by* F, *i.e., one of the first two conditions of Definition 20 fails. Otherwise, we will say that* $\boldsymbol{f}$ *is reducible w.r.t.* F. *If* $\boldsymbol{f} \xrightarrow{F}_+ \boldsymbol{h}$ *and* $\boldsymbol{h}$ *is reduced w.r.t.* F, *then we say that* $\boldsymbol{h}$ *is a remainder for* $\boldsymbol{f}$ *w.r.t.* F.

Remark 21. Related to the previous definition we have the following remarks:
(i) By Theorem 6, the coefficients $c_{\alpha_i,\boldsymbol{f}_i}$ are unique and satisfy

$$x^{\alpha_i}x^{\exp(lm(\boldsymbol{f}_i))} = c_{\alpha_i,\boldsymbol{f}_i}x^{\alpha_i+\exp(lm(\boldsymbol{f}_i))} + p_{\alpha_i,\boldsymbol{f}_i},$$

where $p_{\alpha_i,\boldsymbol{f}_i} = 0$ or $\deg(lm(p_{\alpha_i,\boldsymbol{f}_i})) < |\alpha_i + \exp(lm(\boldsymbol{f}_i))|$, $1 \leq i \leq t$.
(ii) $lm(\boldsymbol{f}) \succ lm(\boldsymbol{h})$ and $\boldsymbol{f} - \boldsymbol{h} \in \langle F \rangle$, where $\langle F \rangle$ is the submodule of A^m generated by F.
(iii) The remainder of $\boldsymbol{f}$ is not unique.
(iv) By definition we will assume that $\boldsymbol{0} \xrightarrow{F} \boldsymbol{0}$.
(v)

$$lt(\boldsymbol{f}) = \sum_{i=1}^{t} r_i lt(x^{\alpha_i} lt(\boldsymbol{f}_i)),$$

The proofs of the next technical proposition and theorem can be also adapted from [5].

Proposition 22. *Let* A *be a* σ*-PBW extension such that* $c_{\alpha,\beta}$ *is invertible for each* $\alpha, \beta \in \mathbb{N}^n$. *Let* $\boldsymbol{f}, \boldsymbol{h} \in A^m$, $\theta \in \mathbb{N}^n$ *and* $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ *be a finite set of non-zero vectors of* A^m. *Then,*

(i) *If* $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$, *then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F} x^\theta \boldsymbol{h}$. *In particular, if* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$.

(ii) *If* $\boldsymbol{f} \xrightarrow{F}_+ \boldsymbol{h}$ *and* $\boldsymbol{p} \in A^m$ *is such that* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(\boldsymbol{h}) \succ lm(\boldsymbol{p})$, *then* $\boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_+ \boldsymbol{h} + \boldsymbol{p}$.

(iii) *If* $\boldsymbol{f} \xrightarrow{F}_+ \boldsymbol{h}$, *then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_+ x^\theta \boldsymbol{h}$. *If* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$.

(iv) *If* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{0}$, *then there exists* $\boldsymbol{p} \in A^m$ *with* $\boldsymbol{p} = \boldsymbol{0}$ *or* $lm(x^\theta \boldsymbol{f}) \succ lm(\boldsymbol{p})$ *such that* $x^\theta \boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{0}$. *If* A *is quasi-commutative, then* $\boldsymbol{p} = \boldsymbol{0}$.

Theorem 23. *Let* $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ *be a set of non-zero vectors of* A^m *and* $\boldsymbol{f} \in A^m$, *then the Division Algorithm below produces polynomials* $q_1, \ldots, q_t \in A$ *and a reduced vector* $\boldsymbol{h} \in A^m$ *w.r.t.* F *such that* $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$ *and*

$$\boldsymbol{f} = q_1 \boldsymbol{f}_1 + \cdots + q_t \boldsymbol{f}_t + \boldsymbol{h}$$

with

$$lm(\boldsymbol{f}) = \max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), \ldots, lm(lm(q_t)lm(\boldsymbol{f}_t)), lm(\boldsymbol{h})\}.$$

Division Algorithm in A^m

INPUT: $\boldsymbol{f}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in A^m$ *with* $\boldsymbol{f}_j \neq \boldsymbol{0}$ $(1 \leq j \leq t)$

OUTPUT: $q_1, \ldots, q_t \in A$, $\boldsymbol{h} \in A^m$ *with* $\boldsymbol{f} = q_1 \boldsymbol{f}_1 + \cdots + q_t \boldsymbol{f}_t + \boldsymbol{h}$, $\boldsymbol{h}$ *reduced w.r.t.* $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ *and*
$lm(\boldsymbol{f}) = \max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), \ldots, lm(lm(q_t)lm(\boldsymbol{f}_t)), lm(\boldsymbol{h})\}$

INITIALIZATION: $q_1 := 0, q_2 := 0, \ldots, q_t := 0, \boldsymbol{h} := \boldsymbol{f}$

WHILE $\boldsymbol{h} \neq \boldsymbol{0}$ *and there exists* j *such that* $lm(\boldsymbol{f}_j)$ *divides* $lm(\boldsymbol{h})$ **DO**

Calculate $J := \{j \mid lm(\boldsymbol{f}_j)$ *divides* $lm(\boldsymbol{h})\}$

FOR $j \in J$ **DO**

Calculate $\alpha_j \in \mathbb{N}^n$ *such that* $\alpha_j + \exp(lm(\boldsymbol{f}_j)) = \exp(lm(\boldsymbol{h}))$

IF *the equation* $lc(\boldsymbol{h}) = \sum_{j \in J} r_j \sigma^{\alpha_j}(lc(\boldsymbol{f}_j)) c_{\alpha_j, \boldsymbol{f}_j}$ *is soluble, where* $c_{\alpha_j, \boldsymbol{f}_j}$ *are defined as in Definition 20*

THEN

Calculate one solution $(r_j)_{j \in J}$

$\boldsymbol{h} := \boldsymbol{h} - \sum_{j \in J} r_j x^{\alpha_j} \boldsymbol{f}_j$

FOR $j \in J$ **DO**

$q_j := q_j + r_j x^{\alpha_j}$

ELSE

Stop

Example 24. We consider the Heisenberg algebra, $A := h_1(2) = \sigma(\mathbb{Q})\langle x, y, z \rangle$, with deglex order and $x > y > z$ in $Mon(A)$ and the TOPREV order in $Mon(A^3)$ with $\boldsymbol{e}_1 \succ \boldsymbol{e}_2 \succ \boldsymbol{e}_3$. Let $\boldsymbol{f} := x^2yz\boldsymbol{e}_1 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + z^2\boldsymbol{e}_3$,

$\boldsymbol{f}_1 := xz\boldsymbol{e}_1 + x\boldsymbol{e}_3 + y\boldsymbol{e}_2$ and $\boldsymbol{f}_2 := xy\boldsymbol{e}_1 + z\boldsymbol{e}_2 + z\boldsymbol{e}_3$. Following the Division Algorithm we will compute $q_1, q_2 \in A$ and $\boldsymbol{h} \in A^3$ such that $\boldsymbol{f} = q_1\boldsymbol{f}_1 + q_2\boldsymbol{f}_2 + \boldsymbol{h}$, with $lm(\boldsymbol{f}) = \max\{lm(lm(q_1)lm(\boldsymbol{f}_1)), lm(lm(q_2)lm(\boldsymbol{f}_2)), lm(\boldsymbol{h})\}$. We will represent the elements of $Mon(A)$ by t^α instead of x^α. For $j = 1, 2$, we will note $\alpha_j := (\alpha_{j1}, \alpha_{j2}, \alpha_{j3}) \in \mathbb{N}^3$.

Step 1: we start with $\boldsymbol{h} := \boldsymbol{f}, q_1 := 0$ and $q_2 := 0$; since $lm(\boldsymbol{f}_1) \mid lm(\boldsymbol{h})$ and $lm(\boldsymbol{f}_2) \mid lm(\boldsymbol{h})$, we compute α_j such that $\alpha_j + \exp(lm(\boldsymbol{f_j})) = \exp(lm(\boldsymbol{h}))$.

. $lm(t^{\alpha_1} lm(\boldsymbol{f}_1)) = lm(\boldsymbol{h})$, so $lm(x^{\alpha_{11}} y^{\alpha_{12}} z^{\alpha_{13}} xz) = x^2yz$, and hence $\alpha_{11} = 1$; $\alpha_{12} = 1$; $\alpha_{13} = 0$. Thus, $t^{\alpha_1} = xy$.

. $lm(t^{\alpha_2} lm(\boldsymbol{f}_2)) = lm(\boldsymbol{h})$, so $lm(x^{\alpha_{21}} y^{\alpha_{22}} z^{\alpha_{23}} xy) = x^2yz$, and hence $\alpha_{21} = 1$; $\alpha_{22} = 0$; $\alpha_{23} = 1$. Thus, $t^{\alpha_2} = xz$.

Next, for $j = 1, 2$ we compute $c_{\alpha_j, \boldsymbol{f_j}}$:

. $t^{\alpha_1} t^{exp(lm(\boldsymbol{f}_1))} = (xy)(xz) = x(2xy)z = 2x^2yz$. Thus, $c_{\alpha_1, \boldsymbol{f}_1} = 2$.

. $t^{\alpha_2} t^{exp(lm(\boldsymbol{f}_2))} = (xz)(xy) = x(\frac{1}{2}xz + y)y = \frac{1}{2}x^2zy + xy^2 = x^2yz + xy^2$. So, $c_{\alpha_2, \boldsymbol{f}_2} = 1$.

We must solve the equation

$$\begin{aligned} 1 = lc(\boldsymbol{h}) &= r_1\sigma^{\alpha_1}(lc(\boldsymbol{f}_1))c_{\alpha_1,\boldsymbol{f}_1} + r_2\sigma^{\alpha_2}(lc(\boldsymbol{f}_2))c_{\alpha_2,\boldsymbol{f}_2} \\ &= r_1\sigma^{\alpha_1}(1)2 + r_2\sigma^{\alpha_2}(1)1 \\ &= 2r_1 + r_2, \end{aligned}$$

then $r_1 = 0$ and $r_2 = 1$.

We make $\boldsymbol{h} := \boldsymbol{h} - (r_1 t^{\alpha_1}\boldsymbol{f}_1 + r_2 t^{\alpha_2}\boldsymbol{f}_2)$, i.e.,

$$\begin{aligned} \boldsymbol{h} &:= \boldsymbol{h} - (xz(xy\boldsymbol{e}_1 + z\boldsymbol{e}_2 + z\boldsymbol{e}_3)) \\ &= \boldsymbol{h} - (xzxy\boldsymbol{e}_1 + xz^2\boldsymbol{e}_2 + xz^2\boldsymbol{e}_3) \\ &= \boldsymbol{h} - ((x^2yz + xy^2)\boldsymbol{e}_1 + xz^2\boldsymbol{e}_2 + xz^2\boldsymbol{e}_3) \\ &= x^2yz\boldsymbol{e}_1 + xz\boldsymbol{e}_1 + y^2z\boldsymbol{e}_2 + z^2\boldsymbol{e}_3 - x^2yz\boldsymbol{e}_1 - xy^2\boldsymbol{e}_1 - xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 \\ &= -xy^2\boldsymbol{e}_1 - xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + z^2\boldsymbol{e}_3. \end{aligned}$$

In addition, we have $q_1 := q_1 + r_1 t^{\alpha_1} = 0$ and $q_2 := q_2 + r_2 t^{\alpha_2} = xz$.

Step 2: $\boldsymbol{h} := -xy^2\boldsymbol{e}_1 - xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + z^2\boldsymbol{e}_3$, so $lm(\boldsymbol{h}) = xy^2\boldsymbol{e}_1$ and $lc(\boldsymbol{h}) = -1$; moreover, $q_1 = 0$ and $q_2 = xz$. Since $lm(\boldsymbol{f}_2) \mid lm(\boldsymbol{h})$, we compute α_2 such that $\alpha_2 + exp(lm(\boldsymbol{f}_2)) = exp(lm(\boldsymbol{h}))$:

. $lm(t^{\alpha_2} lm(\boldsymbol{f}_2)) = lm(\boldsymbol{h})$, then $lm(x^{\alpha_{21}} y^{\alpha_{22}} z^{\alpha_{23}} xy) = xy^2$, so $\alpha_{21} = 0$; $\alpha_{22} = 1$; $\alpha_{23} = 0$. Thus, $t^{\alpha_2} = y$.

We compute $c_{\alpha_2, \boldsymbol{f}_2}$: $t^{\alpha_2} t^{exp(lm(\boldsymbol{f}_2))} = y(xy) = 2xy^2$. Then, $c_{\alpha_2, \boldsymbol{f}_2} = 2$.

We solve the equation

$$\begin{aligned} -1 = lc(\boldsymbol{h}) &= r_2\sigma^{\alpha_2}(lc(\boldsymbol{f}_2))c_{\alpha_2,\boldsymbol{f}_2} \\ &= r_2\sigma^{\alpha_2}(1)2 = 2r_2, \end{aligned}$$

thus, $r_2 = -\frac{1}{2}$.

We make $\boldsymbol{h} := \boldsymbol{h} - r_2 t^{\alpha_2}\boldsymbol{f}_2$, i.e.,

$$\begin{aligned}\boldsymbol{h} &:= \boldsymbol{h} + \frac{1}{2}y(xy\boldsymbol{e}_1 + z\boldsymbol{e}_2 + z\boldsymbol{e}_3)\\ &= \boldsymbol{h} + \frac{1}{2}yxy\boldsymbol{e}_1 + \frac{1}{2}yz\boldsymbol{e}_2 + \frac{1}{2}yz\boldsymbol{e}_3\\ &= -xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + \frac{1}{2}yz\boldsymbol{e}_2 + \frac{1}{2}yz\boldsymbol{e}_3 + z^2\boldsymbol{e}_3.\end{aligned}$$

We have also that $q_1 := 0$ and $q_2 := q_2 + r_2 t^{\alpha_2} = xz - \frac{1}{2}y$.
Step 3: $\boldsymbol{h} = -xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + \frac{1}{2}yz\boldsymbol{e}_2 + \frac{1}{2}yz\boldsymbol{e}_3 + z^2\boldsymbol{e}_3$, so $lm(\boldsymbol{h}) = xz^2\boldsymbol{e}_2$ and $lc(\boldsymbol{h}) = -1$; moreover, $q_1 = 0$ and $q_2 = xz - \frac{1}{2}y$. Since $lm(\boldsymbol{f}_1) \nmid lm(\boldsymbol{h})$ and $lm(\boldsymbol{f}_2) \nmid lm(\boldsymbol{h})$, then $\boldsymbol{h}$ is reduced with respect to $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$, so the algorithm stops.
Thus, we get $q_1, q_2 \in A$ and $\boldsymbol{h} \in A^3$ reduced such that $\boldsymbol{f} = q_1\boldsymbol{f}_1 + q_2\boldsymbol{f}_2 + \boldsymbol{h}$. In fact,

$$\begin{aligned}&q_1\boldsymbol{f}_1 + q_2\boldsymbol{f}_2 + \boldsymbol{h} = 0\boldsymbol{f}_1 + \left(xz - \frac{1}{2}y\right)\boldsymbol{f}_2 + \boldsymbol{h}\\ &= (xz - \frac{1}{2}y)(xy\boldsymbol{e}_1 + z\boldsymbol{e}_2 + z\boldsymbol{e}_3) - xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1\\ &\quad + \frac{1}{2}yz\boldsymbol{e}_2 + \frac{1}{2}yz\boldsymbol{e}_3 + z^2\boldsymbol{e}_3\\ &= x^2yz\boldsymbol{e}_1 + xy^2\boldsymbol{e}_1 - xy^2\boldsymbol{e}_1 + xz^2\boldsymbol{e}_2 - \frac{1}{2}yz\boldsymbol{e}_2 + xz^2\boldsymbol{e}_3 - \frac{1}{2}yz\boldsymbol{e}_3\\ &\quad - xz^2\boldsymbol{e}_2 - xz^2\boldsymbol{e}_3 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + \frac{1}{2}yz\boldsymbol{e}_2 + \frac{1}{2}yz\boldsymbol{e}_3 + z^2\boldsymbol{e}_3\\ &= x^2yz\boldsymbol{e}_1 + y^2z\boldsymbol{e}_2 + xz\boldsymbol{e}_1 + z^2\boldsymbol{e}_3 = \boldsymbol{f},\end{aligned}$$

and $\max\{lm(lm(q_i)lm(\boldsymbol{f}_i)), lm(\boldsymbol{h})\}_{i=1,2} = \max\{0, x^2yz\boldsymbol{e}_1, xz^2\boldsymbol{e}_2\} = x^2yz\boldsymbol{e}_1 = lm(\boldsymbol{f})$.

4 Gröbner bases

Our next purpose is to define Gröbner bases for submodules of A^m.

Definition 25. *Let $M \neq 0$ be a submodule of A^m and let G be a non empty finite subset of non-zero vectors of M, we say that G is a Gröbner basis for M if each element $\mathbf{0} \neq \boldsymbol{f} \in M$ is reducible w.r.t. G.*

We will say that $\{\mathbf{0}\}$ is a Gröbner basis for $M = 0$.

Theorem 26. *Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero vectors of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for M.*

(ii) *For any vector $\boldsymbol{f} \in A^m$,*

$$\boldsymbol{f} \in M \text{ if and only if } \boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{0}.$$

(iii) *For any* $\boldsymbol{0} \neq \boldsymbol{f} \in M$ *there exist* $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ *such that* $lm(\boldsymbol{g}_j)|lm(\boldsymbol{f})$, $1 \leq j \leq t$, *(i.e.,* $ind(lm(\boldsymbol{g}_j)) = ind(lm(\boldsymbol{f}))$ *and there exist* $\alpha_j \in \mathbb{N}^n$ *such that* $\alpha_j + \exp(lm(\boldsymbol{g}_j)) = \exp(lm(\boldsymbol{f})))$ *and*

$$lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\boldsymbol{g}_1}, \dots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\boldsymbol{g}_t} \rangle.$$

(iv) *For* $\alpha \in \mathbb{N}^n$ *and* $1 \leq u \leq m$, *let* $\langle \alpha, M \rangle_u$ *be the left ideal of* R *defined by*

$$\langle \alpha, M \rangle_u := \langle lc(\boldsymbol{f}) | \boldsymbol{f} \in M, ind(lm(\boldsymbol{f})) = u, \exp(lm(\boldsymbol{f})) = \alpha \rangle.$$

Then, $\langle \alpha, M \rangle_u = J_u$, *with*

$$J_u := \langle \sigma^{\beta}(lc(\boldsymbol{g}))c_{\beta,\boldsymbol{g}} | \boldsymbol{g} \in G, ind(lm(\boldsymbol{g})) = u \text{ and } \beta + \exp(lm(\boldsymbol{g})) = \alpha \rangle.$$

Proof. (i)⇒ (ii): let $\boldsymbol{f} \in M$, if $\boldsymbol{f} = \boldsymbol{0}$, then by definition $\boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{0}$. If $\boldsymbol{f} \neq \boldsymbol{0}$, then there exists $\boldsymbol{h}_1 \in A^m$ such that $\boldsymbol{f} \xrightarrow{G} \boldsymbol{h}_1$, with $lm(\boldsymbol{f}) \succ lm(\boldsymbol{h}_1)$ and $\boldsymbol{f} - \boldsymbol{h}_1 \in \langle G \rangle \subseteq M$, hence $\boldsymbol{h}_1 \in M$; if $\boldsymbol{h}_1 = \boldsymbol{0}$, so we end. If $\boldsymbol{h}_1 \neq \boldsymbol{0}$, then we can repeat this reasoning for $\boldsymbol{h}_1$, and since $Mon(A^m)$ is well ordered, we get that $\boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{0}$.
Conversely, if $\boldsymbol{f} \xrightarrow{G}_{+} \boldsymbol{0}$, then by Theorem 23, there exist $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ and $q_1, \dots, q_t \in A$ such that $\boldsymbol{f} = q_1\boldsymbol{g}_1 + \dots + q_t\boldsymbol{g}_t$, i.e., $\boldsymbol{f} \in M$.
(ii)⇒ (i): evident.
(i)⇔ (iii): this is a direct consequence of Definition 20.
(iii)⇒ (iv) Since R is left Noetherian, there exist $r_1, \dots, r_s \in R$, $\boldsymbol{f}_1, \dots, \boldsymbol{f}_l \in M$ such that $\langle \alpha, M \rangle_u = \langle r_1, \dots, r_s \rangle$, $ind(lm(\boldsymbol{f}_i)) = u$ and $\exp(lm(\boldsymbol{f}_i)) = \alpha$ for each $1 \leq i \leq l$, with $\langle r_1, \dots, r_s \rangle \subseteq \langle lc(\boldsymbol{f}_1), \dots, lc(\boldsymbol{f}_l) \rangle$. Then, $\langle lc(\boldsymbol{f}_1), \dots, lc(\boldsymbol{f}_l) \rangle = \langle \alpha, M \rangle_u$. Let $r \in \langle \alpha, M \rangle_u$, there exist $a_1, \dots, a_l \in R$ such that $r = a_1 lc(\boldsymbol{f}_1) + \dots + a_l lc(\boldsymbol{f}_l)$; by (iii), for each i, $1 \leq i \leq l$, there exist $\boldsymbol{g}_{1i}, \dots, \boldsymbol{g}_{t_i i} \in G$ and $b_{ji} \in R$ such that $lc(\boldsymbol{f}_i) = b_{1i}\sigma^{\alpha_{1i}}(lc(\boldsymbol{g}_{1i}))c_{\alpha_{1i},\boldsymbol{g}_{1i}} + \dots + b_{t_i i}\sigma^{\alpha_{t_i i}}(lc(\boldsymbol{g}_{t_i i}))c_{\alpha_{t_i i},\boldsymbol{g}_{t_i i}}$, with $u = ind(lm(\boldsymbol{f}_i)) = ind(lm(\boldsymbol{g}_{ji}))$ and $\exp(lm(\boldsymbol{f}_i)) = \alpha_{ji} + \exp(lm(\boldsymbol{g}_{ji}))$, thus $\langle \alpha, M \rangle_u \subseteq J_u$. Conversely, if $r \in J_u$, then $r = b_1\sigma^{\beta_1}(lc(\boldsymbol{g}_1))c_{\beta_1,\boldsymbol{g}_1} + \dots + b_t\sigma^{\beta_t}(lc(\boldsymbol{g}_t))c_{\beta_t,\boldsymbol{g}_t}$, with $b_i \in R$, $\beta_i \in \mathbb{N}^n$, $\boldsymbol{g}_i \in G$ such that $ind(lm(\boldsymbol{g}_i)) = u$ and $\beta_i + \exp(lm(\boldsymbol{g}_i)) = \alpha$ for any $1 \leq i \leq t$. Note that $x^{\beta_i}\boldsymbol{g}_i \in M$, $ind(lm(x^{\beta_i}\boldsymbol{g}_i)) = u$, $\exp(lm(x^{\beta_i}\boldsymbol{g}_i)) = \alpha$, $lc(x^{\beta_i}\boldsymbol{g}_i) = \sigma^{\beta_i}(lc(\boldsymbol{g}_i))c_{\beta_i,\boldsymbol{g}_i}$, for $1 \leq i \leq t$, and $r = b_1 lc(x^{\beta_1}\boldsymbol{g}_1) + \dots + b_t lc(x^{\beta_t}\boldsymbol{g}_t)$, i.e., $r \in \langle \alpha, M \rangle_u$.
(iv)⇒ (iii): let $\boldsymbol{0} \neq \boldsymbol{f} \in M$ and let $u = ind(lm(\boldsymbol{f}))$, $\alpha = \exp(lm(\boldsymbol{f}))$, then $lc(\boldsymbol{f}) \in \langle \alpha, M \rangle_u$; by (iv) $lc(\boldsymbol{f}) = b_1\sigma^{\beta_1}(lc(\boldsymbol{g}_1))c_{\beta_1,\boldsymbol{g}_1} + \dots + b_t\sigma^{\beta_t}(lc(\boldsymbol{g}_t))c_{\beta_t,\boldsymbol{g}_t}$, with $b_i \in R$, $\beta_i \in \mathbb{N}^n$, $\boldsymbol{g}_i \in G$ such that $u = ind(lm(\boldsymbol{g}_i))$ and $\beta_i + \exp(lm(\boldsymbol{g}_i)) = \alpha$ for any $1 \leq i \leq t$. From this we conclude that $lm(\boldsymbol{g}_j)|lm(\boldsymbol{f})$, $1 \leq j \leq t$. □

From this theorem we get the following consequences.

Corollary 27. *Let* $M \neq 0$ *be a submodule of* A^m. *Then,*

(i) *If G is a Gröbner basis for M, then $M = \langle G\rangle$.*

(ii) *Let G be a Gröbner basis for M, if $\boldsymbol{f} \in M$ and $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{h}$, with $\boldsymbol{h}$ reduced w.r.t. G, then $\boldsymbol{h} = \boldsymbol{0}$.*

(iii) *Let $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a set of non-zero vectors of M with $lc(\boldsymbol{g}_i) = 1$, for each $1 \leq i \leq t$, such that given $\boldsymbol{r} \in M$ there exists i such that $lm(\boldsymbol{g}_i)$ divides $lm(\boldsymbol{r})$. Then, G is a Gröbner basis of M.*

5 Computing Gröbner bases

The following two theorems are the support for the Buchberger's algorithm for computing Gröbner bases when A is a quasi-commutative bijective $\sigma - PBW$ extension The proofs of these results are as in [5].

Definition 28. *Let $F := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\} \subseteq A^m$ such that the least common multiple of $\{lm(\boldsymbol{g}_1), \ldots, lm(\boldsymbol{g}_s)\}$, denoted by $\boldsymbol{X}_F$, is non-zero. Let $\theta \in \mathbb{N}^n$, $\beta_i := \exp(lm(\boldsymbol{g}_i))$ and $\gamma_i \in \mathbb{N}^n$ such that $\gamma_i + \beta_i = \exp(\boldsymbol{X}_F)$, $1 \leq i \leq s$. $B_{F,\theta}$ will denote a finite set of generators of*

$$S_{F,\theta} := Syz_R[\sigma^{\gamma_1+\theta}(lc(\boldsymbol{g}_1))c_{\gamma_1+\theta,\beta_1} \ \cdots \ \sigma^{\gamma_s+\theta}(lc(\boldsymbol{g}_s))c_{\gamma_s+\theta,\beta_s})].$$

For $\theta = \boldsymbol{0} := (0, \ldots, 0)$, $S_{F,\theta}$ will be denoted by S_F and $B_{F,\theta}$ by B_F.

Theorem 29. *Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero generators of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of M.*

(ii) *For all $F := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\} \subseteq G$, with $\boldsymbol{X}_F \neq \boldsymbol{0}$, and for all $\theta \in \mathbb{N}^n$ and any $(b_1, \ldots, b_s) \in B_{F,\theta}$,*

$$\textstyle\sum_{i=1}^s b_i x^{\gamma_i+\theta}\boldsymbol{g}_i \xrightarrow{G}_+ 0.$$

In particular, if G is a Gröbner basis of M then for all $F := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\} \subseteq G$, with $\boldsymbol{X}_F \neq \boldsymbol{0}$, and any $(b_1, \ldots, b_s) \in B_F$,

$$\textstyle\sum_{i=1}^s b_i x^{\gamma_i}\boldsymbol{g}_i \xrightarrow{G}_+ 0.$$

Theorem 30. *Let A be a quasi-commutative bijective $\sigma - PBW$ extension. Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero generators of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of M.*

(ii) *For all $F := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\} \subseteq G$, with $\boldsymbol{X}_F \neq \boldsymbol{0}$, and any $(b_1, \ldots, b_s) \in B_F$,*

$$\textstyle\sum_{i=1}^s b_i x^{\gamma_i}\boldsymbol{g}_i \xrightarrow{G}_+ \boldsymbol{0}.$$

Corollary 31. *Let A be a quasi-commutative bijective $\sigma - PBW$ extension. Let $F = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}$ be a set of non-zero vectors of A^m. The algorithm below produces a Gröbner basis for the submodule $\langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s\rangle$ ($P(X)$ denotes the set of subsets of the set X):*

Gröbner Basis Algorithm for Modules
over Quasi-Commutative Bijective $\sigma - PBW$ Extensions

INPUT: $F := \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\} \subseteq A^m$, $\boldsymbol{f}_i \neq \boldsymbol{0}$, $1 \leq i \leq s$

OUTPUT: $G = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$ *a Gröbner basis for* $\langle F \rangle$

INITIALIZATION: $G := \emptyset, G' := F$

WHILE $G' \neq G$ **DO**

$D := P(G') - P(G)$

$G := G'$

FOR *each* $S := \{\boldsymbol{g}_{i_1}, \dots, \boldsymbol{g}_{i_k}\} \in D$, *with* $\boldsymbol{X}_S \neq \boldsymbol{0}$, **DO**

Compute B_S

FOR *each* $\boldsymbol{b} = (b_1, \dots, b_k) \in B_S$ **DO**

Reduce $\sum_{j=1}^{k} b_j x^{\gamma_j} \boldsymbol{g}_{i_j} \xrightarrow{G'}_{+} \boldsymbol{r}$, *with* $\boldsymbol{r}$ *reduced with respect to* G' *and* γ_j *defined as in Definition 28*

IF $\boldsymbol{r} \neq \boldsymbol{0}$ **THEN**

$G' := G' \cup \{\boldsymbol{r}\}$

From Theorem 8 and the previous corollary we get the following direct conclusion.

Corollary 32. *Let A be a quasi-commutative bijective $\sigma - PBW$ extension. Then each submodule of A^m has a Gröbner basis.*

Now we will illustrate with an example the algorithm presented in Corollary 31.

Example 33. We will consider the multiplicative analogue of the Weyl algebra

$$A := \mathcal{O}_3(\lambda_{21}, \lambda_{31}, \lambda_{32}) = \mathcal{O}_3\left(2, \frac{1}{2}, 3\right) = \sigma(\mathbb{Q}[x_1])\langle x_2, x_3\rangle,$$

hence we have the relations

$$x_2x_1 = \lambda_{21}x_1x_2 = 2x_1x_2, \text{ so } \sigma_2(x_1) = 2x_1 \text{ and } \delta_2(x_1) = 0,$$

$$x_3x_1 = \lambda_{31}x_1x_3 = \frac{1}{2}x_1x_3, \text{ so } \sigma_3(x_1) = \frac{1}{2}x_1 \text{ and } \delta_3(x_1) = 0,$$

$$x_3x_2 = \lambda_{32}x_2x_3 = 3x_2x_3, \text{ so } c_{2,3} = 3,$$

and for $r \in \mathbb{Q}$, $\sigma_2(r) = r = \sigma_3(r)$. We choose in $Mon(A)$ the deglex order with $x_2 > x_3$ and in $Mon(A^2)$ the TOPREV order with $\boldsymbol{e}_1 \succ \boldsymbol{e}_2$.
Let $\boldsymbol{f}_1 = x_1^2x_2^2\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2$, $lm(\boldsymbol{f}_1) = x_2^2\boldsymbol{e}_1$ and $\boldsymbol{f}_2 = 2x_1x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $lm(\boldsymbol{f}_2) = x_2x_3\boldsymbol{e}_1$. We will construct a Gröbner basis for the module $M := \langle \boldsymbol{f}_1, \boldsymbol{f}_2\rangle$.
Step 1: we start with $G := \emptyset$, $G' := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. Since $G' \neq G$, we make $D := \mathcal{P}(G') - \mathcal{P}(G) = \{S_1, S_2, S_{1,2}\}$, with $S_1 := \{\boldsymbol{f}_1\}, S_2 := \{\boldsymbol{f}_2\}, S_{1,2} := \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. We also make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we compute B_S:
. For S_1 we have

$$Syz_{\mathbb{Q}[x_1]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}],$$

where $\beta_1 = \exp(lm(\boldsymbol{f}_1)) = (2,0)$; $\boldsymbol{X}_{S_1} = l.c.m.\{lm(\boldsymbol{f}_1)\} = lm(\boldsymbol{f}_1) = x_2^2\boldsymbol{e}_1$; $\exp(\boldsymbol{X}_{S_1}) = (2,0)$; $\gamma_1 = \exp(\boldsymbol{X}_{S_1}) - \beta_1 = (0,\ 0)$; $x^{\gamma_1}x^{\beta_1} = x_2^2$, so $c_{\gamma_1,\beta_1} = 1$. Then,

$$\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1} = \sigma^{\gamma_1}(x_1^2)1 = \sigma_2^0\sigma_3^0(x_1^2) = x_1^2.$$

Thus, $Syz_{\mathbb{Q}[x_1]}[x_1^2] = \{0\}$ and $B_{S_1} = \{0\}$, i.e., we do not add any vector to G'.
. For S_2 we have an identical situation.
. For $S_{1,2}$ we compute

$$Syz_{\mathbb{Q}[x_1]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1} \quad \sigma^{\gamma_2}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2}],$$

where $\beta_1 = \exp(lm(\boldsymbol{f}_1)) = (2,0)$ and $\beta_2 = \exp(lm(\boldsymbol{f}_2)) = (1,1)$;
$\boldsymbol{X}_{S_{1,2}} = l.c.m.\{lm(\boldsymbol{f}_1), lm(\boldsymbol{f}_2)\} = l.c.m.(x_2^2\boldsymbol{e}_1, x_2x_3\boldsymbol{e}_1) = x_2^2x_3\boldsymbol{e}_1$;
$\exp(\boldsymbol{X}_{S_{1,2}}) = (2,1)$; $\gamma_1 = \exp(\boldsymbol{X}_{S_{1,2}}) - \beta_1 = (0,\ 1)$ and $\gamma_2 = \exp(\boldsymbol{X}_{S_{1,2}}) - \beta_2 = (1,\ 0)$; $x^{\gamma_1}x^{\beta_1} = x_3x_2^2 = 3x_2x_3x_2 = 9x_2^2x_3$, so $c_{\gamma_1,\beta_1} = 9$; in a similar way $x^{\gamma_2}x^{\beta_2} = x_2^2x_3$, i.e., $c_{\gamma_2,\beta_2} = 1$. Then,

$$\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1} = \sigma^{\gamma_1}(x_1^2)9 = \sigma_2^0\sigma_3(x_1^2)9 = (\sigma_3(x_1)\sigma_3(x_1))9 = \frac{9}{4}x_1^2$$

and

$$\sigma^{\gamma_2}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2} = \sigma^{\gamma_2}(2x_1)1 = \sigma_2\sigma_3^0(2x_1) = \sigma_2(2x_1) = 4x_1.$$

Hence $Syz_{\mathbb{Q}[x_1]}[\frac{9}{4}x_1^2 \quad 4x_1] = \{(b_1, b_2) \in \mathbb{Q}[x_1]^2 \mid b_1(\frac{9}{4}x_1^2) + b_2(4x_1) = 0\}$ and $B_{S_{1,2}} = \{(4, -\frac{9}{4}x_1)\}$. From this we get

$$\begin{aligned}
4x^{\gamma_1}\boldsymbol{f}_1 - \frac{9}{4}x_1x^{\gamma_2}\boldsymbol{f}_2 &= 4x_3(x_1^2x_2^2\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2) - \frac{9}{4}x_1x_2(2x_1x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)\\
&= 4x_3x_1^2x_2^2\boldsymbol{e}_1 + 4x_3x_2x_3\boldsymbol{e}_2 - \frac{9}{4}x_1x_22x_1x_2x_3\boldsymbol{e}_1 - \frac{9}{4}x_1x_2^2\boldsymbol{e}_2\\
&= 9x_1^2x_2^2x_3\boldsymbol{e}_1 + 12x_2x_3^2\boldsymbol{e}_2 - 9x_1^2x_2^2x_3\boldsymbol{e}_1 - \frac{9}{4}x_1x_2^2\boldsymbol{e}_2\\
&= 12x_2x_3^2\boldsymbol{e}_2 - \frac{9}{4}x_1x_2^2\boldsymbol{e}_2 := \boldsymbol{f}_3,
\end{aligned}$$

so $lm(\boldsymbol{f}_3) = x_2x_3^2\boldsymbol{e}_2$. We observe that $\boldsymbol{f}_3$ is reduced with respect to G'. We make $G' := G' \cup \{\boldsymbol{f}_3\}$, i.e., $G' = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$.
Step 2: since $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2\} \neq G' = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$, we make $D := \mathcal{P}(G') - \mathcal{P}(G)$, i.e., $D := \{S_3, S_{1,3}, S_{2,3}, S_{1,2,3}\}$, where $S_1 := \{\boldsymbol{f}_1\}, S_{1,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_3\}, S_{2,3} := \{\boldsymbol{f}_2, \boldsymbol{f}_3\}, S_{1,2,3} := \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$. We make $G := G'$, and for every $S \in D$ such that $\boldsymbol{X}_S \neq \boldsymbol{0}$ we must compute B_S. Since $\boldsymbol{X}_{S_{1,3}} = \boldsymbol{X}_{S_{2,3}} = \boldsymbol{X}_{S_{1,2,3}} = \boldsymbol{0}$, we only need to consider S_3.
. We have to compute

$$Syz_{\mathbb{Q}[x_1]}[\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3}],$$

where $\beta_3 = \exp(lm(\boldsymbol{f}_3)) = (1,2)$; $\boldsymbol{X}_{S_3} = l.c.m.\{lm(\boldsymbol{f}_3)\} = lm(\boldsymbol{f}_3) = x_2x_3^2\boldsymbol{e}_2$; $\exp(\boldsymbol{X}_{S_3}) = (1,2)$; $\gamma_3 = \exp(\boldsymbol{X}_{S_3}) - \beta_3 = (0,0)$; $x^{\gamma_3}x^{\beta_3} = x_2x_3^2$, so $c_{\gamma_3,\beta_3} = 1$. Hence

$$\sigma^{\gamma_3}(lc(\boldsymbol{f}_3))c_{\gamma_3,\beta_3} = \sigma^{\gamma_3}(12)1 = \sigma_2^0\sigma_3^0(12) = 12,$$

and $Syz_{\mathbb{Q}[x_1]}[12] = \{0\}$, i.e., $B_{S_3} = \{0\}$. This means that we not add any vector to G' and hence $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a Gröbner basis for M.

6 Syzygy of a module

We present in this section a method for computing the syzygy module of a submodule $M = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s \rangle$ of A^m using Gröbner bases. This implies that we have a method for computing such bases. Thus, we will assume that A is a bijective quasi-commutative σ-PBW extension.
Let f be the canonical homomorphism defined by

$$\begin{aligned} A^s &\xrightarrow{f} A^m \\ \boldsymbol{e}_j &\mapsto \boldsymbol{f}_j \end{aligned}$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_s\}$ is the canonical basis of A^s. Observe that f can be represented by a matrix, i.e., if $\boldsymbol{f}_j := (f_{1j}, \dots, f_{mj})^T$, then the matrix of f in the canonical bases of A^s and A^m is

$$F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1s} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{ms} \end{bmatrix} \in M_{m\times s}(A).$$

Note that $Im(f)$ is the column module of F, i.e., the left A-module generated by the columns of F:

$$Im(f) = \langle f(\boldsymbol{e}_1), \dots, f(\boldsymbol{e}_s) \rangle = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s \rangle = \langle F \rangle.$$

Moreover, observe that if $\boldsymbol{a} := (a_1, \dots, a_s)^T \in A^s$, then

$$f(\boldsymbol{a}) = (\boldsymbol{a}^T F^T)^T. \tag{6.1}$$

In fact,

$$\begin{aligned}
f(\boldsymbol{a}) &= a_1 f(\boldsymbol{e}_1) + \cdots + a_s f(\boldsymbol{e}_s) = a_1 \boldsymbol{f}_1 + \cdots + a_s \boldsymbol{f}_s \\
&= a_1 \begin{bmatrix} f_{11} \\ \vdots \\ f_{m1} \end{bmatrix} + \cdots + a_s \begin{bmatrix} f_{1s} \\ \vdots \\ f_{ms} \end{bmatrix} \\
&= \begin{bmatrix} a_1 f_{11} + \cdots + a_s f_{1s} \\ \vdots \\ a_1 f_{m1} + \cdots + a_s f_{ms} \end{bmatrix} \\
&= (\begin{bmatrix} a_1 & \cdots & a_s \end{bmatrix} \begin{bmatrix} f_{11} & \cdots & f_{m1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{ms} \end{bmatrix})^T \\
&= (\boldsymbol{a}^T F^T)^T.
\end{aligned}$$

We recall that

$$Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) := \{\boldsymbol{a} := (a_1, \ldots, a_s)^T \in A^s | a_1 \boldsymbol{f}_1 + \cdots + a_s \boldsymbol{f}_s = \boldsymbol{0}\}.$$

Note that

$$Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) = \ker(f), \tag{6.2}$$

but $Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) \neq \ker(F)$ since we have

$$\boldsymbol{a} \in Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}) \Leftrightarrow \boldsymbol{a}^T F^T = \boldsymbol{0}. \tag{6.3}$$

The modules of syzygies of M and F are defined by

$$Syz(M) := Syz(F) := Syz(\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}). \tag{6.4}$$

The generators of $Syz(F)$ can be disposed into a matrix, so sometimes we will refer to $Syz(F)$ as a matrix. Thus, if $Syz(F)$ is generated by r vectors, $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$, then

$$Syz(F) = \langle \boldsymbol{z}_1, \ldots, \boldsymbol{z}_r \rangle,$$

and we will use the following matrix notation

$$Syz(F) := Z(F) := \begin{bmatrix} \boldsymbol{z}_1 & \cdots & \boldsymbol{z}_r \end{bmatrix} = \begin{bmatrix} z_{11} & \cdots & z_{1r} \\ \vdots & & \vdots \\ z_{s1} & \cdots & z_{sr} \end{bmatrix} \in M_{s \times r}(A),$$

thus we have

$$Z(F)^T F^T = 0. \tag{6.5}$$

Let $G := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a Gröbner basis of M, then from Division Algorithm and Corollary 27, there exist polynomials $q_{ij} \in A$, $1 \leq i \leq t$, $1 \leq j \leq s$ such

that

$$
\begin{aligned}
\boldsymbol{f}_1 &= q_{11}\boldsymbol{g}_1 + \cdots + q_{t1}\boldsymbol{g}_t \\
&\vdots \\
\boldsymbol{f}_s &= q_{1s}\boldsymbol{g}_1 + \cdots + q_{ts}\boldsymbol{g}_t,
\end{aligned}
$$

i.e.,

$$F^T = Q^T G^T, \tag{6.6}$$

with

$$Q := [q_{ij}] = \begin{bmatrix} q_{11} & \cdots & q_{1s} \\ \vdots & & \vdots \\ q_{t1} & \cdots & q_{ts} \end{bmatrix}, \; G := \begin{bmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_t \end{bmatrix} := \begin{bmatrix} g_{11} & \cdots & g_{1t} \\ \vdots & & \vdots \\ g_{m1} & \cdots & g_{mt} \end{bmatrix}.$$

From (6.6) we get

$$Z(F)^T Q^T G^T = 0. \tag{6.7}$$

From the algorithm of Corollary 31 we observe that each element of G can be expressed as an A-linear combination of columns of F, i.e., there exists polynomials $h_{ji} \in A$ such that

$$
\begin{aligned}
\boldsymbol{g}_1 &= h_{11}\boldsymbol{f}_1 + \cdots + h_{s1}\boldsymbol{f}_s \\
&\vdots \\
\boldsymbol{g}_t &= h_{1t}\boldsymbol{f}_1 + \cdots + h_{st}\boldsymbol{f}_s,
\end{aligned}
$$

so we have

$$G^T = H^T F^T, \tag{6.8}$$

with

$$H := [h_{ji}] = \begin{bmatrix} h_{11} & \cdots & h_{1t} \\ \vdots & & \vdots \\ h_{s1} & \cdots & h_{st} \end{bmatrix}.$$

The next theorem will prove that $Syz(F)$ can be calculated using $Syz(G)$, and in turn, Lemma 39 below will establish that for quasi-commutative bijective $\sigma - PBW$ extensions, $Syz(G)$ can be computed using $Syz(L_G)$, where

$$L_G := \begin{bmatrix} lt(\boldsymbol{g}_1) & \cdots & lt(\boldsymbol{g}_t) \end{bmatrix}.$$

Suppose that $Syz(L_G)$ is generated by l elements,

$$Syz(L_G) := Z(L_G) := \begin{bmatrix} \boldsymbol{z}''_1 & \cdots & \boldsymbol{z}''_l \end{bmatrix} = \begin{bmatrix} z''_{11} & \cdots & z''_{1l} \\ \vdots & & \vdots \\ z''_{t1} & \cdots & z''_{tl} \end{bmatrix}. \tag{6.9}$$

The proof of Lemma 39 will show that $Syz(G)$ can be generated also by l elements, say, $\boldsymbol{z}'_1, \ldots, \boldsymbol{z}'_l$, i.e., $Syz(G) = \langle \boldsymbol{z}'_1, \ldots, \boldsymbol{z}'_l \rangle$; we write

$$Syz(G) := Z(G) := \begin{bmatrix} \boldsymbol{z}'_1 & \cdots & \boldsymbol{z}'_l \end{bmatrix} = \begin{bmatrix} z'_{11} & \cdots & z'_{1l} \\ \vdots & & \vdots \\ z'_{t1} & \cdots & z'_{tl} \end{bmatrix} \in M_{t\times l}(A),$$

and hence

$$Z(G)^T G^T = 0. \tag{6.10}$$

Theorem 34. *With the above notation, $Syz(F)$ coincides with the column module of the extended matrix $\begin{bmatrix}(Z(G)^T H^T)^T & I_s - (Q^T H^T)^T\end{bmatrix}$, i.e., in a matrix notation*

$$Syz(F) = \begin{bmatrix}(Z(G)^T H^T)^T & I_s - (Q^T H^T)^T\end{bmatrix}. \tag{6.11}$$

Proof. Let $\boldsymbol{z} := (z_1, \ldots, z_s)^T$ be one of generators of $Syz(F)$, i.e., one of columns of $Z(F)$, then by (6.3) $\boldsymbol{z}^T F^T = \boldsymbol{0}$, and by (6.6) we have $\boldsymbol{z}^T Q^T G^T = \boldsymbol{0}$. Let $\boldsymbol{u} := (\boldsymbol{z}^T Q^T)^T$, then $\boldsymbol{u} \in Syz(G)$ and there exists polynomials $w_1, \ldots, w_l \in A$ such that $\boldsymbol{u} = w_1 \boldsymbol{z}'_1 + \cdots + w_l \boldsymbol{z}'_l$, i.e., $\boldsymbol{u} = (\boldsymbol{w}^T Z(G)^T)^T$, with $\boldsymbol{w} := (w_1, \ldots, w_l)^T$. Then, $\boldsymbol{u}^T H^T = (\boldsymbol{w}^T Z(G)^T) H^T$, i.e., $\boldsymbol{z}^T Q^T H^T = (\boldsymbol{w}^T Z(G)^T) H^T$ and from this we have

$$\begin{aligned}\boldsymbol{z}^T &= \boldsymbol{z}^T Q^T H^T + \boldsymbol{z}^T - \boldsymbol{z}^T Q^T H^T \\ &= \boldsymbol{z}^T Q^T H^T + \boldsymbol{z}^T (I_s - Q^T H^T) \\ &= (\boldsymbol{w}^T Z(G)^T) H^T + \boldsymbol{z}^T (I_s - Q^T H^T).\end{aligned}$$

From this can be checked that $\boldsymbol{z} \in \langle \begin{bmatrix}(Z(G)^T H^T)^T & I_s - (Q^T H^T)^T\end{bmatrix} \rangle$.
Conversely, from (6.8) and (6.10) we have $(Z(G)^T H^T) F^T = Z(G)^T (H^T F^T) = Z(G)^T G^T = 0$, but this means that each column of $(Z(G)^T H^T)^T$ is in $Syz(F)$. In a similar way, from (6.8) and (6.6) we get $(I_s - Q^T H^T) F^T = F^T - Q^T H^T F^T = F^T - Q^T G^T = F^T - F^T = 0$, i.e., each column of $(I_s - Q^T H^T)^T$ is also in $Syz(F)$. This complete the proof. □

Our next task is to compute $Syz(L_G)$. Let $L = [c_1 \boldsymbol{X}_1 \cdots c_t \boldsymbol{X}_t]$ be a matrix of size $m \times t$, where $\boldsymbol{X}_1 = X_1 \boldsymbol{e}_{i_1}, \ldots, \boldsymbol{X}_t = X_t \boldsymbol{e}_{i_t}$ are monomials of A^m, $c_1, \ldots, c_t \in A - \{0\}$ and $1 \leq i_1, \ldots, i_t \leq m$. We note that some indexes $i_1, \ldots, i_t$ could be equals.

Definition 35. *We say that a syzygy $\boldsymbol{h} = (h_1, \ldots, h_t)^T \in Syz(L)$ is homogeneous of degree $\boldsymbol{X} = X \boldsymbol{e}_i$, where $X \in Mon(A)$ and $1 \leq i \leq m$, if*

(i) *h_j is a term, for each $1 \leq j \leq t$.*

(ii) *For each $1 \leq j \leq t$, either $h_j = 0$ or if $h_j \neq 0$ then $lm(lm(h_j)\boldsymbol{X}_j) = \boldsymbol{X}$.*

Proposition 36. *Let L be as above. For quasi-commutative $\sigma - PBW$ extensions, $Syz(L)$ has a finite generating set of homogeneous syzygies.*

Proof. Since A^t is a Noetherian module, $Syz(L)$ is a finitely generated submodule of A^t. So, it is enough to prove that each generator $\boldsymbol{h} = (h_1, \ldots, h_t)^T$ of $Syz(L)$ is a finite sum of homogeneous syzygies of $Syz(L)$. We have $h_1 c_1 X_1 \boldsymbol{e}_{i_1} +$

$\cdots + h_t c_t X_t \boldsymbol{e}_{i_t} = \boldsymbol{0}$, and we can group together summands according to equal canonical vectors such that $\boldsymbol{h}$ can be expressed as a finite sum of syzygies of $Syz(L)$. We observe that each of such syzygies have null entries for those places j where $\boldsymbol{e}_{i_j}$ does not coincide with the canonical vector of its group. The idea is to prove that each of such syzygies is a sum of homogeneous syzygies of $Syz(L)$. But this means that we have reduced the problem to Lemma 4.2.2 of [1], where the canonical vector is the same for all entries. We include the proof for completeness.
So, let $\boldsymbol{f} = (f_1, \ldots, f_t)^T \in Syz(c_1X_1, \ldots, c_tX_t)$, then $f_1c_1X_1 + \cdots + f_tc_tX_t = 0$; we expand each polynomial f_j as a sum of u terms (adding zero summands, if it is necessary):

$$f_j = a_{1j}Y_1 + \cdots + a_{uj}Y_u,$$

where $a_{lj} \in R$ and $Y_1 \succ Y_2 \succ \cdots \succ Y_u \in Mon(A)$ are the different monomials we found in $f_1, \ldots, f_t$, $1 \leq j \leq t$. Then,

$$(a_{11}Y_1 + \cdots + a_{u1}Y_u)c_1X_1 + \cdots + (a_{1t}Y_1 + \cdots + a_{ut}Y_u)c_tX_t = 0.$$

Since A is quasi-commutative, the product of two terms is a term, so in the previous relation we can assume that there are $d \leq tu$ different monomials, $Z_1, \ldots, Z_d$. Hence, completing with zero entries (if it is necessary), we can write

$$\boldsymbol{f} = (b_{11}Y_{11}, \ldots, b_{1t}Y_{1t})^T + \cdots + (b_{d1}Y_{d1}, \ldots, b_{dt}Y_{dt})^T,$$

where $(b_{k1}Y_{k1}, \ldots, b_{kt}Y_{kt})^T \in Syz(c_1X_1, \ldots, c_tX_t)$ is homogeneous of degree Z_k, $1 \leq k \leq d$. $\square$

Definition 37. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t \in Mon(A^m)$ and let $J \subseteq \{1, \ldots, t\}$. Let*

$$\boldsymbol{X}_J = lcm\{\boldsymbol{X}_j | j \in J\}.$$

We say that J is saturated with respect to $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t\}$, if

$$\boldsymbol{X}_j | \boldsymbol{X}_J \Rightarrow j \in J,$$

for any $j \in \{1, \ldots, t\}$. The saturation J' of J consists of all $j \in \{1, \ldots, t\}$ such that $\boldsymbol{X}_j | \boldsymbol{X}_J$.

Lemma 38. *Let L be as above. For quasi-commutative bijective $\sigma - PBW$ extensions, a homogeneous generating set for $Syz(L)$ is*

$$\{\boldsymbol{s}_v^J | J \subseteq \{1, \ldots, t\} \text{ is saturated with respect to } \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t\}, 1 \leq v \leq r_J\},$$

where

$$\boldsymbol{s}_v^J = \sum_{j \in J} b_{vj}^J x^{\gamma_j} \boldsymbol{e}_j,$$

with $\gamma_j \in \mathbb{N}^n$ such that $\gamma_j + \beta_j = \exp(\boldsymbol{X}_J)$, $\beta_j = \exp(\boldsymbol{X}_j)$, $j \in J$, and $\boldsymbol{b}_v^J := (b_{vj}^J)_{j \in J}$, with $B^J := \{\boldsymbol{b}_1^J, \ldots, \boldsymbol{b}_{r_J}^J\}$ is a set of generators for $Syz_R[\sigma^{\gamma_j}(c_j)c_{\gamma_j, \beta_j} \mid j \in J]$.

Proof. First note that $\boldsymbol{s}_v^J$ is a homogeneous syzygy of $Syz(L)$ of degree $\boldsymbol{X}_J$ since each entry of $\boldsymbol{s}_v^J$ is a term, for each non-zero entry we have $lm(x^{\gamma_j}\boldsymbol{X}_j) = \boldsymbol{X}_J$, and moreover, if $i_J := ind(\boldsymbol{X}_J)$, then

$$\begin{aligned}((\boldsymbol{s}_v^J)^T L^T)^T &= \sum_{j\in J} b_{vj}^J x^{\gamma_j} c_j \boldsymbol{X}_j = \sum_{j\in J} b_{vj}^J \sigma^{\gamma_j}(c_j) x^{\gamma_j} \boldsymbol{X}_j \\ &= (\sum_{j\in J}(b_{vj}^J \sigma^{\gamma_j}(c_j) c_{\gamma_j,\beta_j}) x^{\gamma_j+\beta_j})\boldsymbol{e}_{i_J} \\ &= \boldsymbol{0}.\end{aligned}$$

On the other hand, let $\boldsymbol{h} \in Syz(L)$, then by Proposition 36, $Syz(L)$ is generated by homogeneous syzygies, so we can assume that $\boldsymbol{h}$ is a homogeneous syzygy of some degree $\boldsymbol{Y} = Y\boldsymbol{e}_i$, $Y := x^{\alpha}$. We will represent $\boldsymbol{h}$ as a linear combination of syzygies of type $\boldsymbol{s}_v^J$. Let $\boldsymbol{h} = (d_1Y_1, \ldots, d_tY_t)^T$, with $d_k \in R$ and $Y_k := x^{\alpha_k}$, $1 \le k \le t$, let $J = \{j \in \{1, \ldots, t\} | d_j \neq 0\}$, then $lm(Y_j\boldsymbol{X}_j) = \boldsymbol{Y}$ for $j \in J$, and

$$\boldsymbol{0} = \sum_{j\in J} d_j Y_j c_j \boldsymbol{X}_j = \sum_{j\in J} d_j \sigma^{\alpha_j}(c_j) Y_j \boldsymbol{X}_j = \sum_{j\in J} d_j \sigma^{\alpha_j}(c_j) c_{\alpha_j,\beta_j} \boldsymbol{Y}.$$

In addition, since $lm(Y_j\boldsymbol{X}_j) = \boldsymbol{Y}$ then $\boldsymbol{X}_j \mid \boldsymbol{Y}$ for any $j \in J$, and hence $\boldsymbol{X}_J \mid \boldsymbol{Y}$, i.e., there exists θ such that $\theta + \exp(\boldsymbol{X}_J) = \alpha = \theta + \gamma_j + \beta_j$; but, $\alpha_j + \beta_j = \alpha$ since $lm(Y_j\boldsymbol{X}_j) = \boldsymbol{Y}$, so $\alpha_j = \theta + \gamma_j$.
Thus,

$$\boldsymbol{0} = \sum_{j\in J} d_j \sigma^{\alpha_j}(c_j) c_{\alpha_j,\beta_j} \boldsymbol{Y} = \sum_{j\in J} d_j \sigma^{\theta+\gamma_j}(c_j) c_{\theta+\gamma_j,\beta_j} \boldsymbol{Y},$$

and from Remark 7 we get that

$$\begin{aligned}0 &= \sum_{j\in J} d_j \sigma^{\theta+\gamma_j}(c_j) c_{\theta+\gamma_j,\beta_j} = \sum_{j\in J} d_j c_{\theta,\gamma_j}^{-1} c_{\theta,\gamma_j} \sigma^{\theta+\gamma_j}(c_j) c_{\theta+\gamma_j,\beta_j} \\ &= \sum_{j\in J} d_j c_{\theta,\gamma_j}^{-1} \sigma^{\theta}(\sigma^{\gamma_j}(c_j)) c_{\theta,\gamma_j} c_{\theta+\gamma_j,\beta_j} \\ &= \sum_{j\in J} d_j c_{\theta,\gamma_j}^{-1} \sigma^{\theta}(\sigma^{\gamma_j}(c_j)) \sigma^{\theta}(c_{\gamma_j,\beta_j}) c_{\theta,\gamma_j+\beta_j}.\end{aligned}$$

We multiply the last equality by $c_{\theta,\exp(\boldsymbol{X}_J)}^{-1}$, but $c_{\theta,\exp(\boldsymbol{X}_J)}^{-1} = c_{\theta,\gamma_j+\beta_j}^{-1}$ for any $j \in J$, so

$$0 = \sum_{j\in J} d_j c_{\theta,\gamma_j}^{-1} \sigma^{\theta}(\sigma^{\gamma_j}(c_j) c_{\gamma_j,\beta_j}).$$

Since A is bijective, there exists d_j' such that $\sigma^{\theta}(d_j') = d_j c_{\theta,\gamma_j}^{-1}$, so

$$0 = \sum_{j\in J} \sigma^{\theta}(d_j') \sigma^{\theta}(\sigma^{\gamma_j}(c_j) c_{\gamma_j,\beta_j}),$$

and from this we get

$$0 = \sum_{j\in J} d'_j \sigma^{\gamma_j}(c_j)c_{\gamma_j,\beta_j}.$$

Let J' be the saturation of J with respect to $\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_t\}$, since $d_j = 0$ if $j \in J' - J$, then $d'_j = 0$, and hence, $(d'_j \mid j \in J') \in Syz_R[\sigma^{\gamma_j}(c_j)c_{\gamma_j,\beta_j} \mid j \in J']$. From this we have

$$(d'_j \mid j \in J') = \sum_{v=1}^{r_{J'}} a_v b^{J'}_{vj}.$$

Since $\boldsymbol{X}_{J'} = \boldsymbol{X}_J$, then $\boldsymbol{X}_{J'}$ also divides $\boldsymbol{Y}$, and hence

$$\begin{aligned}
\boldsymbol{h} &= \sum_{j=1}^{t} d_j Y_j \boldsymbol{e}_j = \sum_{j\in J'} d_j c^{-1}_{\theta,\gamma_j} x^{\theta} x^{\gamma_j} \boldsymbol{e}_j = \sum_{j\in J'} \sigma^{\theta}(d'_j) x^{\theta} x^{\gamma_j} \boldsymbol{e}_j \\
&= \sum_{j\in J'} x^{\theta} d'_j x^{\gamma_j} \boldsymbol{e}_j = \sum_{j\in J'} x^{\theta} \left(\sum_{v=1}^{r_{J'}} a_v b^{J'}_{vj}\right) x^{\gamma_j} \boldsymbol{e}_j = \sum_{j\in J'} \sum_{v=1}^{r_{J'}} x^{\theta} a_v b^{J'}_{vj} x^{\gamma_j} \boldsymbol{e}_j \\
&= \sum_{v=1}^{r_{J'}} x^{\theta} a_v \sum_{j\in J'} b^{J'}_{vj} x^{\gamma_j} \boldsymbol{e}_j \\
&= \sum_{v=1}^{r_{J'}} \sigma^{\theta}(a_v) x^{\theta} \boldsymbol{s}^{J'}_v.
\end{aligned}$$

□

Finally, we will calculate $Syz(G)$ using $Syz(L_G)$. Applying Division Algorithm and Corollary 27 to the columns of $Syz(L_G)$ (see (6.9)), for each $1 \leq v \leq l$ there exists polynomials $p_{1v}, \dots, p_{tv} \in A$ such that

$$z''_{1v}\boldsymbol{g}_1 + \cdots + z''_{tv}\boldsymbol{g}_t = p_{1v}\boldsymbol{g}_1 + \cdots + p_{tv}\boldsymbol{g}_t,$$

i.e.,

$$Z(L_G)^T G^T = P^T G^T, \tag{6.12}$$

with

$$P := \begin{bmatrix} p_{11} & \cdots & p_{1l} \\ \vdots & & \vdots \\ p_{t1} & \cdots & p_{tl} \end{bmatrix}.$$

With this notation, we have the following result.

Lemma 39. *For quasi-commutative bijective $\sigma - PBW$ extensions, the column module of $Z(G)$ coincides with the column module of $Z(L_G) - P$, i.e., in a matrix notation*

$$Z(G) = Z(L_G) - P. \tag{6.13}$$

Proof. From (6.12), $(Z(L_G) - P)^T G^T = 0$, so each column of $Z(L_G) - P$ is in $Syz(G)$, i.e., each column of $Z(L_G) - P$ is an A-linear combination of columns of $Z(G)$. Thus, $\langle Z(L_G) - P\rangle \subseteq \langle Z(G)\rangle$.
Now we have to prove that $\langle Z(G)\rangle \subseteq \langle Z(L_G) - P\rangle$. Suppose that $\langle Z(G)\rangle \nsubseteq \langle Z(L_G)-P\rangle$, so there exists $\boldsymbol{z'} = (z'_1, \ldots, z'_t)^T \in \langle Z(G)\rangle$ such that $\boldsymbol{z'} \notin \langle Z(L_G)-P\rangle$; from all such vectors we choose one such that

$$\boldsymbol{X} := \max_{1\leq j\leq t}\{lm(lm(z'_j)lm(\boldsymbol{g}_j))\} \tag{6.14}$$

be the least. Let $\boldsymbol{X} = X\boldsymbol{e}_i$ and

$$J := \{j \in \{1, \ldots, t\} | lm(lm(z'_j)lm(\boldsymbol{g}_j)) = \boldsymbol{X}\}.$$

Since A is quasi-commutative and $\boldsymbol{z'} \in Syz(G)$ then

$$\sum_{j\in J} lt(z'_j)lt(\boldsymbol{g}_j) = \boldsymbol{0}.$$

Let $\boldsymbol{h} := \sum_{j\in J} lt(z'_j)\widetilde{\boldsymbol{e}}_j$, where $\widetilde{\boldsymbol{e}}_1, \ldots, \widetilde{\boldsymbol{e}}_t$ is the canonical basis of A^t. Then, $\boldsymbol{h} \in Syz(lt(\boldsymbol{g}_1), \ldots, lt(\boldsymbol{g}_t))$ is a homogeneous syzygy of degree $\boldsymbol{X}$. Let $B := \{\boldsymbol{z}''_1, \ldots, \boldsymbol{z}''_l\}$ be a homogeneous generating set for the syzygy module $Syz(L_G))$, where $\boldsymbol{z}''_v$ has degree $\boldsymbol{Z}_v = Z_v\boldsymbol{e}_{i_v}$ (see (6.9)). Then, $\boldsymbol{h} = \sum_{v=1}^{l} a_v\boldsymbol{z}''_v$, where $a_v \in A$, and hence

$$\boldsymbol{h} = (a_1z''_{11} + \cdots + a_lz''_{1l}, \ldots, a_1z''_{t1} + \cdots + a_lz''_{tl})^T.$$

We can assume that for each $1 \leq v \leq l$, a_v is a term. In fact, consider the first entry of $\boldsymbol{h}$: completing with null terms, each a_v is an ordered sum of s terms

$$(c_{11}X_{11} + \cdots + c_{1s}X_{1s})z''_{11} + \cdots + (c_{l1}X_{l1} + \cdots + c_{ls}X_{ls})z''_{1l},$$

with $X_{v1} \succ X_{v2} \succ \cdots \succ X_{vs}$ for each $1 \leq v \leq l$, so

$$\left.\begin{array}{c} lm(X_{11}lm(z''_{11})) \succ lm(X_{12}lm(z''_{11})) \succ \cdots \succ lm(X_{1s}lm(z''_{11})) \\ \vdots \\ lm(X_{l1}lm(z''_{1l})) \succ lm(X_{l2}lm(z''_{1l})) \succ \cdots \succ lm(X_{ls}lm(z''_{1l})) \end{array}\right\} \tag{6.15}$$

Since each $\boldsymbol{z}''_v$ is a homogeneous syzygy, each entry z''_{jv} of $\boldsymbol{z}''_v$ is a term, but the first entry of $\boldsymbol{h}$ is also a term, then from (6.15) we can assume that a_v is a term. We note that for $j \in J$

$$lt(z'_j) = a_1z''_{j1} + \cdots + a_lz''_{jl},$$

and for $j \notin J$

$$a_1z''_{j1} + \cdots + a_lz''_{jl} = 0.$$

Moreover, let $j \in J$, so $lm(lm(a_1z''_{j1}+\cdots+a_lz''_{jl})lm(\boldsymbol{g}_j)) = lm(lm(z'_j)lm(\boldsymbol{g}_j)) = \boldsymbol{X}$, and we can choose those v such that $lm(a_vz''_{jv}) = lm(z'_j)$, for the others v we can take $a_v = 0$. Thus, for j and such v we have

$$lm(lm(a_v)lm(lm(z''_{jv})lm(\boldsymbol{g}_j))) = \boldsymbol{X} = X\boldsymbol{e}_i.$$

On the other hand, for $j, j' \in J$ with $j' \neq j$, we know that $\boldsymbol{z}''_v$ is homogeneous of degree $\boldsymbol{Z}_v = Z_v \boldsymbol{e}_{i_v}$, hence, if $z''_{j'v} \neq 0$, then $lm(lm(z''_{j'v})lm(\boldsymbol{g}_{j'})) = \boldsymbol{Z}_v = lm(lm(z''_{jv})lm(\boldsymbol{g}_j))$. Thus, we must conclude that $i_v = i$ and

$$lm(lm(a_v)lm(lm(z''_{jv})lm(\boldsymbol{g}_j))) = \boldsymbol{X}, \tag{6.16}$$

for any v and any j such that $a_v \neq 0$ and $z''_{jv} \neq 0$.
We define $\boldsymbol{q}' := (q'_1, \ldots, q'_t)^T$, where $q'_j := z'_j$ if $j \notin J$ and $q'_j := z'_j - lt(z'_j)$ if $j \in J$. We observe that $\boldsymbol{z}' = \boldsymbol{h} + \boldsymbol{q}'$, and hence $\boldsymbol{z}' = \sum_{v=1}^{l} a_v \boldsymbol{z}''_v + \boldsymbol{q}' = \sum_{v=1}^{l} a_v(\boldsymbol{s}_v + \boldsymbol{p}_v) + \boldsymbol{q}'$, with $\boldsymbol{s}_v := \boldsymbol{z}''_v - \boldsymbol{p}_v$, where $\boldsymbol{p}_v$ is the column v of matrix P defined in (6.12). Then, we define

$$\boldsymbol{r} := (\textstyle\sum_{v=1}^{l} a_v \boldsymbol{p}_v) + \boldsymbol{q}',$$

and we note that $\boldsymbol{r} = \boldsymbol{z}' - \sum_{v=1}^{l} a_v \boldsymbol{s}_v \in Syz(G) - \langle Z(L_G) - P \rangle$. We will get a contradiction proving that $\max_{1 \le j \le t}\{lm(lm(r_j)lm(\boldsymbol{g}_j))\} \prec \boldsymbol{X}$. For each $1 \le j \le t$ we have

$$r_j = a_1 p_{j1} + \cdots + a_l p_{jl} + q'_j$$

and hence

$$\begin{aligned} lm(lm(r_j)lm(\boldsymbol{g}_j)) &= lm(lm(a_1 p_{j1} + \cdots + a_l p_{jl} + q'_j)lm(\boldsymbol{g}_j)) \\ &\preceq lm(\max\{lm(a_1 p_{j1} + \cdots + a_l p_{jl}), lm(q'_j)\}lm(\boldsymbol{g}_j)) \\ &\preceq lm(\max\{\max_{1 \le v \le l}\{lm(lm(a_v)lm(p_{jv}))\}, lm(q'_j)\}lm(\boldsymbol{g}_j)). \end{aligned}$$

By the definition of $\boldsymbol{q}'$ we have that for each $1 \le j \le t$, $lm(lm(q'_j)lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}$. In fact, if $j \notin J$, $lm(lm(q'_j)lm(\boldsymbol{g}_j)) = lm(lm(z'_j)lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}$, and for $j \in J$, $lm(lm(q'_j)lm(\boldsymbol{g}_j)) = lm(lm(z'_j - lt(z'_j))lm(\boldsymbol{g}_j)) \prec \boldsymbol{X}$. On the other hand,

$$\sum_{j=1}^{t} z''_{jv} \boldsymbol{g}_j = \sum_{j=1}^{t} p_{jv} \boldsymbol{g}_j,$$

with

$$lm(\sum_{j=1}^{t} z''_{jv} \boldsymbol{g}_j) = \max_{1 \le j \le t}\{lm(lm(p_{jv})lm(\boldsymbol{g}_j))\}.$$

But, $\sum_{j=1}^{t} z''_{jv} lt(\boldsymbol{g}_j) = \boldsymbol{0}$ for each v, then

$$lm(\sum_{j=1}^{t} z''_{jv} \boldsymbol{g}_j) \prec \max_{1 \le j \le t}\{lm(lm(z''_{jv})lm(\boldsymbol{g}_j))\}.$$

Hence,

$$\max_{1 \le j \le t}\{lm(lm(p_{jv})lm(\boldsymbol{g}_j))\} \prec \max_{1 \le j \le t}\{lm(lm(z''_{jv})lm(\boldsymbol{g}_j))\}$$

for each $1 \leq v \leq l$. From (6.16), $\max_{\substack{1\leq j\leq t\\ 1\leq v\leq l}}\{lm(lm(a_v)lm(lm(p_{jv})lm(\boldsymbol{g}_j)))\} \prec \max_{\substack{1\leq j\leq t\\ 1\leq v\leq l}}\{lm(lm(a_v)lm(lm(z''_{jv})lm(\boldsymbol{g}_j)))\} = \boldsymbol{X}$, and hence, we can conclude that $\max_{1\leq j\leq t}\{lm(lm(r_j)lm(\boldsymbol{g}_j))\} \prec \boldsymbol{X}$. $\square$

Example 40. Let $M := \langle \boldsymbol{f}_1, \boldsymbol{f}_2\rangle$, where $\boldsymbol{f}_1 = x_1^2x_2^2\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2$ and $\boldsymbol{f}_2 = 2x_1x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \in A^2$, with $A := \sigma(\mathbb{Q}[x_1])\langle x_2, x_3\rangle$. In Example 33 we computed a Gröbner basis $G = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ of M, where $\boldsymbol{f}_3 = 12x_2x_3^2\boldsymbol{e}_2 - \frac{9}{4}x_1x_2^2\boldsymbol{e}_2$. Now we will calculate $Syz(F)$ with $F = \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$:

(i) Firstly we compute $Syz(L_G)$ using Lemma 38:

$$L_G := \begin{bmatrix} lt(\boldsymbol{f}_1) & lt(\boldsymbol{f}_2) & lt(\boldsymbol{f}_3)\end{bmatrix} = \begin{bmatrix} x_1^2x_2^2\boldsymbol{e}_1 & 2x_1x_2x_3\boldsymbol{e}_1 & 12x_2x_3^2\boldsymbol{e}_2\end{bmatrix}.$$

For this we choose the saturated subsets J of $\{1, 2, 3\}$ with respect to $\{x_2^2\boldsymbol{e}_1, x_2x_3\boldsymbol{e}_1, x_2x_3^2\boldsymbol{e}_2\}$ and such that $\boldsymbol{X}_J \neq 0$:

- For $J_1 = \{1\}$ we compute a system B^{J_1} of generators of

$$Syz_{\mathbb{Q}[x_1]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1}],$$

where $\beta_1 := \exp(lm(\boldsymbol{f}_1))$ and $\gamma_1 = \exp(\boldsymbol{X}_{J_1}) - \beta_1$. Then, $B^{J_1} = \{0\}$, and hence we have only one generator $\boldsymbol{b}_1^{J_1} = (b_{11}^{J_1}) = 0$ and $\boldsymbol{s}_1^{J_1} = b_{11}^{J_1}x^{\gamma_1}\tilde{\boldsymbol{e}}_1 = 0\tilde{\boldsymbol{e}}_1$, with $\tilde{\boldsymbol{e}}_1 = (1, 0, 0)^T$.

- For $J_2 = \{2\}$ and $J_3 = \{3\}$ the situation is similar.
- For $J_{1,2} = \{1, 2\}$, a system of generators of

$$Syz_{\mathbb{Q}[x_1]}[\sigma^{\gamma_1}(lc(\boldsymbol{f}_1))c_{\gamma_1,\beta_1} \quad \sigma^{\gamma_2}(lc(\boldsymbol{f}_2))c_{\gamma_2,\beta_2}],$$

where $\beta_1 = \exp(lm(\boldsymbol{f}_1))$, $\beta_2 = \exp(lm(\boldsymbol{f}_2))$, $\gamma_1 = \exp(\boldsymbol{X}_{J_{1,2}}) - \beta_1$ and $\gamma_2 = \exp(\boldsymbol{X}_{J_{1,2}}) - \beta_2$, is $B^{J_{1,2}} = \{(4, -\frac{9}{4}x_1)\}$, thus we have only one generator $\boldsymbol{b}_1^{J_{1,2}} = (b_{11}^{J_{1,2}}, b_{12}^{J_{1,2}}) = (4, -\frac{9}{4}x_1)$ and

$$\begin{aligned}\boldsymbol{s}_1^{J_{1,2}} &= b_{11}^{J_{1,2}}x^{\gamma_1}\tilde{\boldsymbol{e}}_1 + b_{12}^{J_{1,2}}x^{\gamma_2}\tilde{\boldsymbol{e}}_2\\ &= 4x_3\tilde{\boldsymbol{e}}_1 - \frac{9}{4}x_1x_2\tilde{\boldsymbol{e}}_2\\ &= \begin{pmatrix} 4x_3\\ -\frac{9}{4}x_1x_2\\ 0\end{pmatrix}.\end{aligned}$$

Then,

$$Syz(L_G) = \left\langle \begin{pmatrix} 4x_3\\ -\frac{9}{4}x_1x_2\\ 0\end{pmatrix}\right\rangle,$$

or in a matrix notation

$$Syz(L_G) = Z(L_G) = \begin{bmatrix} 4x_3\\ -\frac{9}{4}x_1x_2\\ 0\end{bmatrix}.$$

(ii) Next we compute $Syz(G)$: By Division Algorithm we have

$$4x_3\boldsymbol{f}_1 - \frac{9}{4}x_1x_2\boldsymbol{f}_2 + 0\boldsymbol{f}_3 = p_{11}\boldsymbol{f}_1 + p_{21}\boldsymbol{f}_2 + p_{31}\boldsymbol{f}_3,$$

so by the Example 33, $p_{11} = 0 = p_{21}$ and $p_{31} = 1$, i.e., $P = \tilde{\boldsymbol{e}}_3$. Thus,

$$\begin{aligned} Z(G) &= Z(L_G) - P \\ &= \begin{bmatrix} 4x_3 \\ -\frac{9}{4}x_1x_2 \\ -1 \end{bmatrix} \end{aligned}$$

and

$$Syz(G) = \left\langle \begin{pmatrix} 4x_3 \\ -\frac{9}{4}x_1x_2 \\ -1 \end{pmatrix} \right\rangle.$$

(iii) Finally we compute $Syz(F)$: since

$$\boldsymbol{f}_1 = 1\boldsymbol{f}_1 + 0\boldsymbol{f}_2 + 0\boldsymbol{f}_3,\ \boldsymbol{f}_2 = 0\boldsymbol{f}_1 + 1\boldsymbol{f}_2 + 0\boldsymbol{f}_3$$

then

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Moreover,

$$\boldsymbol{f}_1 = 1\boldsymbol{f}_1 + 0\boldsymbol{f}_2,\ \boldsymbol{f}_2 = 0\boldsymbol{f}_1 + 1\boldsymbol{f}_2,\ \boldsymbol{f}_3 = 4x_3\boldsymbol{f}_1 - \frac{9}{4}x_1x_2\boldsymbol{f}_2,$$

hence

$$H = \begin{bmatrix} 1 & 0 & 4x_3 \\ 0 & 1 & -\frac{9}{4}x_1x_2 \end{bmatrix}.$$

By Theorem 34,

$$Syz(F) = \begin{bmatrix} (Z(G)^T H^T)^T & I_2 - (Q^T H^T)^T \end{bmatrix},$$

with

$$\begin{aligned} (Z(G)^T H^T)^T &= \left(\begin{bmatrix} 4x_3 & -\frac{9}{4}x_1x_2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4x_3 & -\frac{9}{4}x_1x_2 \end{bmatrix} \right)^T \\ &= \left(\begin{bmatrix} 0 & 0 \end{bmatrix} \right)^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

and

$$I_2 - (Q^T H^T)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From this we conclude that $Syz(F) = 0$. Observe that this means that M is free.

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